Teaching Matters and So Does Curriculum: How CUNY Start Reshaped Instruction for Students Referred to Developmental Mathematics

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Abstract

Adult proficiency in numeracy in the United States lags that of other developed nations, and the nonselective institutions that dominate the higher education sector struggle to address the learning needs of the sizeable proportion of students who enroll in their institutions and are deemed academically underprepared in mathematics. Research on curriculum and pedagogy in developmental (or remedial) mathematics indicates that typical teaching approaches emphasize memorization, often at the expense of the kinds of conceptual understanding that prepare students for college-level mathematics and the numeracy demands of the workforce. This paper examines CUNY Start, an innovative pre-matriculation developmental education program developed by The City University of New York (CUNY) that reimagines the design and implementation of remedial instruction to better serve students with weak academic preparation.

Using data from interviews, classroom observations, an instructor survey, and curricular materials, this paper describes four key features of the CUNY Start mathematics instructional approach, paying particular attention to how these features differ from traditional developmental education. These features are: (1) the use of a highly detailed curricular document as a primary resource for instructors; (2) an emphasis on real-world contexts and number relationships, which serve as the instructional starting point (rather than rules and procedures); (3) a pedagogical approach that elicits student talk and discussion through questioning; and (4) explicit attention to students' organizational and study skills. This paper also elaborates on the processes, structures, and resources built into CUNY Start that support its implementation.

This paper is part of an ongoing random assignment evaluation of CUNY Start undertaken with MDRC that so far finds that the program has significant positive effects on students achieving college readiness in mathematics (longer-term effects will also be estimated). This evidence strongly suggests that CUNY Start's structures, processes, and resources enable instructors to teach mathematics in a different way that may boost student achievement.

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1. Introduction

Numeracy is "the ability to access, use, interpret, and communicate mathematical information and ideas in order to engage in and manage the mathematical demands of a range of situations in adult life" (OECD, 2016, p. 48). According to results of the Organization for Economic Cooperation and Development (OECD) Survey of Adult Skills,[1] adult proficiency in numeracy in the United States lags that of other developed nations. The mean proficiency score of 16–65 year-olds in numeracy in the United States is well below the average among OECD countries/economies. These scores are consequential for individuals' life outcomes: Higher proficiency levels are associated with higher rates of employment and earnings and indicators of individual well-being, such as good health and civic participation (OECD, 2016).

The underlying reasons for the nation's comparative weakness in numeracy, or mathematical proficiency, are many and complex. The U.S. secondary education system is generating more high school graduates, including more graduates from historically disadvantaged populations, yet the percentage of ACT-tested high school graduates meeting ACT college readiness benchmarks in mathematics has declined from 46 percent in 2012 to 41 percent in 2016 (ACT, 2016). The nonselective institutions that dominate the higher education sector seek to preserve and expand access; however, they have had limited success in effectively remediating the many students—including the nearly three out of every five community college students—who enroll and are deemed academically underprepared in mathematics.[2] These disappointing results—amid gains in high school completion and college enrollment—point to potential limitations in, among other areas, traditional instructional and curricular approaches to mathematics.

Within higher education, mathematics curriculum and instruction have come under increased scrutiny as questions have been raised about the extent to which students

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[1] The Survey of Adult Skills is conducted by the OECD Programme for the International Assessment of Adult Competencies (PIAAC) and is designed to measure adults' proficiency in literacy, numeracy, and problem solving in technology-rich environments in 33 countries/economies.

[2] Among students who began college in 2003-04, 59 percent of those who started in public two-year colleges and 33 percent of those who started at public four-year colleges took one or more remedial math courses. Half of those at the two-year colleges and 58 percent of those at the four-year colleges completed their remedial mathematics courses (Chen, 2016; see also Bailey, Jeong, & Cho, 2010; Ganga, Mazzariello, & Edgecombe, 2018).

develop critical thinking and other skills through their academic course-taking at college (Arum & Roksa, 2010). At the developmental mathematics level, research indicates that dominant teaching approaches rely on memorization, which can serve to undermine the transfer and application of mathematical knowledge to new contexts (Givvin, Stigler, & Thompson, 2011; Grubb & Gabriner, 2013). Studies also suggest that the lack of emphasis on conceptual understanding in developmental mathematics may leave students unprepared for college-level mathematics (Quarles & Davis, 2017; Richland, Stigler, & Holyoak, 2012). Mathematics educators studying calculus have long questioned both students' preparation for the course and the course itself, given its low pass rates (Steen, 1988). The Mathematics Association of America is currently conducting a study of the precalculus through calculus II sequence at U.S. colleges and universities to better understand the relationship between structural, curricular, and pedagogical factors and student success (Bressoud, Mesa, & Rasmussen, 2015). Addressing these and other challenges to mathematics curriculum and instruction is extremely challenging given the deeply embedded norms and expectations that govern what and how instructors teach. Scholarship on the topic suggests that teaching is an entrenched cultural practice that is resistant, though not impervious, to change (Stigler & Hiebert, 1999).

This paper examines CUNY Start, an innovative pre-matriculation developmental education[3] program developed by The City University of New York (CUNY) that purposefully reimagines the design and implementation of remedial instruction to better serve students with weak academic preparation. CUNY Start provides instruction in reading, writing, and/or mathematics, as well as academic and nonacademic support services, through an intensive, one-semester intervention that students deemed not college-ready complete before formally matriculating in a CUNY college. This paper, which focuses on the program's mathematical approach, is part of a larger U.S. Department of Education Institute of Education Sciences (IES)-funded evaluation of CUNY Start. The evaluation is undertaken through a researcher-practitioner partnership

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[3] In this paper, we use the words "developmental education" and "remediation" interchangeably. In the context of CUNY, these terms refer to courses and interventions in mathematics, reading, and writing that students who fail to achieve specified proficiency scores on the CUNY Assessment Tests or who are not exempted based on high school Regents tests and SAT scores must take before they are permitted to enroll in introductory college-level mathematics, English, and, in some instances, other disciplinary courses.

involving MDRC,[4] CUNY, and the Community College Research Center (CCRC), and is focused on the implementation and outcomes of CUNY Start (Scrivener et al., 2018; Scrivener & Logue, 2016).

In this paper, we will answer the following research questions:

1. What are the key features of the CUNY Start mathematics instructional approach?

2. How does the CUNY Start mathematics instructional approach differ from traditional developmental mathematics instructional approaches?

3. What types of programmatic elements did CUNY Start build to ensure its instructors were able to implement its instructional approach?

The CUNY Start evaluation collected a broad array of implementation data at four CUNY community colleges. This paper draws on these data sources, including interviews and focus groups with 156 instructors, tutors, professional developers, and students; 52 CUNY Start and non-CUNY Start classroom observations; an instructor survey; and curricular materials. See Appendix A, Data and Methods, for a detailed summary of data collection activities, sample information, and analysis techniques. See Appendix B for excerpts from CUNY Start's detailed curriculum document.

2. An Overview of CUNY Start

CUNY Start was developed as a community college-based pre-matriculation program to help students from High School Equivalency (HSE) programs[5] and with low levels of mathematics and literacy proficiency strengthen their skills and, ultimately, be more successful in college. Professional development coordinators in the CUNY Adult Literacy/HSE program launched the precursor to CUNY Start, the College Transition Program (CTP), in 2007 with redesigned mathematics and reading and writing classes.

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[4] MDRC is a nonprofit, nonpartisan education and social policy research organization.
[5] These were formerly known as General Education Development (GED) programs; for ease of exposition, we refer to all such programs in any time period as HSE.

The impetus for the new classes was the staff developers' concern about the large proportion of HSE completers placing into developmental education courses, particularly elementary algebra on the mathematics side, and struggling in CUNY academic programs (Hinds, 2009). For more than two years, they experimented with CTP curricula and instruction, eventually arriving at an innovative model that substantially increased weekly instructional hours, ordered and integrated content into curriculum documents used in very similar ways by all instructors, and incorporated a variety of advisement resources. CTP, which eventually became known as CUNY Start,[6] aspired to not only reduce or eliminate students' need for developmental education but to do so while deepening students' understanding of mathematics and literacy and preparing them to successfully matriculate into college and apply that learning to coursework in CUNY degree programs (Hinds, 2009).

CUNY Start operates at 10 CUNY colleges and is centrally administered by the CUNY Office of Academic Affairs (OAA). At each college that offers the program, CUNY Start is managed by program directors and coordinators and staffed with mathematics and reading and writing instructors,[7] advisors, and academic support personnel. In 2009, CUNY Start expanded its participant pool to include students with traditional high school diplomas (in addition to HSE completers). While it started as a pilot program at Kingsborough Community College and LaGuardia Community College, CUNY Start has since expanded to operate at all seven CUNY community colleges, along with Medgar Evers College, New York City College of Technology, and College of Staten Island.[8] Importantly, the one-semester program costs students only $75, which allows them to preserve their financial aid eligibility. See Scrivener et al. (2018) for a detailed report on CUNY Start, the IES-sponsored evaluation, program implementation, and results on short-term effects from a randomized controlled trial.

The CUNY Start instructional approach to mathematics uses student-centered teaching to engage students in a deep exploration of mathematical content. Scrivener et

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[6] The College Transition Program was renamed the College Transition Initiative (CTI) in 2009. The CTI was eventually renamed CUNY Start.

[7] CUNY Start is taught by full-time Continuing Education instructors who are eligible to receive health and other employment benefits.

[8] Gutmann Community College and New York City College of Technology offer only Math Start, described below.

al. (2018) describe CUNY Start's student-centered mathematics instruction as positioning students "as active participants in their own learning, given time to think and struggle, and encouraged to speak and respond to each other," (p. 29). The CUNY Start instructional approach is demanding for both students and instructors and shaped the program's design. The mathematics component comprises about 11 hours of instructional time per week.[9] Embedded mathematics tutors are available to provide student support both during class time and outside of class. Since the CUNY Start instructional approach diverges from how mathematics is normally taught, CUNY Start professional developers created and continue to refine multi-faceted professional learning resources for instructors. CUNY Start mathematics and reading and writing instructors work closely with academic advisors throughout the semester to support and monitor student performance and quickly identify and address academic and nonacademic obstacles that may impede students' progress. Advisors teach a weekly seminar for students focused on college success skills and resources. They also meet individually with students who are identified as struggling by instructors.

The evidence regarding the effectiveness of CUNY Start, while still emerging, is promising and has driven its expansion within CUNY. The program served 150 students in 2009; it served 1,659 in fall 2018. Moreover, at the colleges where the program is offered, CUNY Start has emerged as the primary remediation strategy for students deemed academically underprepared in all three subject areas: mathematics, reading, and writing. Academic advisors are expected to recommend CUNY Start for these students. Students with one or two remedial needs may be referred to the University Skills Immersion Program (USIP), which provides pre-term tuition-free intensive remedial instruction, or other developmental education offerings (though they are not blocked from enrolling full- or part-time in CUNY Start). On the heels of its success in mathematics, CUNY Start launched Math Start, an eight-week version of its semester-long intensive mathematics program. CUNY Start and Math Start are both direct pipelines to CUNY's renowned Accelerated Study in Associate Programs (ASAP), which has consistently

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[9] The integrated reading and writing component requires about 11 hours of instructional time per week as well. Full-time CUNY Start students attend the program for up to 25 hours per week, which includes a weekly student success seminar. CUNY Start also offers a part-time program in which students can enroll in either mathematics or reading and writing. See Scrivener et al. (2018) for more information.

doubled the graduation rate of participating students since 2007 and is in the midst of a system-wide scale up effort (Strumbos, Linderman, & Hicks, 2018).

3. CUNY Start and the Developmental Mathematics Reform Landscape

Within the broader developmental mathematics reform landscape, CUNY Start makes two main contributions. First, it is one of few remedial programs explicitly designed to serve students with extremely weak academic preparation. Many developmental mathematics reforms have instituted prerequisite requirements, essentially shutting out students who fail to demonstrate proficiency in arithmetic and pre-algebra. The field has clamored for better alternatives for the lowest-placed students, even articulating this priority in guidelines for the field (Strong Start to Finish, 2015). Second, CUNY Start re-envisions teaching and learning in order to strengthen students' mathematics proficiency. Program developers carefully detailed student learning goals and created the curriculum and model structure to achieve those goals. They assumed (and the evidence appears to bear out) that their focus on high quality mathematics instruction would allow more students to not only meet the CUNY proficiency standards in mathematics but also successfully matriculate and persist in college.

Other developmental mathematics reform models do not ignore curriculum and instruction. Corequisite remediation, in which developmental students co-enroll in an introductory college mathematics and a support course, has catalyzed experimentation regarding the ways academic support can be structured to increase the likelihood of success in the college course (Logue, Watanabe-Rose, & Douglas, 2016). Mathematics pathways reforms, which accelerate developmental students' entry into college mathematics courses aligned with their programs of study, have triggered conversations about what types of and how many mathematics courses different degree programs require (Hayward & Willett, 2014; Sowers & Yamada, 2015; Zachry Rutschow & Diamond, 2015). Notably, the mathematics pathways movement has embraced an emphasis on the psychosocial dimensions of learning early on in its development, which has shaped curriculum and its delivery (Silva & White, 2013). Yet neither these reforms nor other noteworthy changes to the traditional developmental education system (such as

using multiple measures for more accurate placement of students) have the explicit focus on teaching and learning of CUNY Start.

In the next section we describe the key features of the CUNY Start mathematics instructional approach, providing detail of its design and implementation, as well as perspectives of instructors and students. We also elaborate on the processes, structures, and resources built into CUNY Start that support its implementation.

4. Features of the CUNY Start Mathematics Instructional Approach

Drawing on the review of the CUNY Start curriculum, observations of CUNY Start classrooms, instructor survey data, and interviews with professional development coordinators and CUNY Start instructors and students, we describe four interconnected features of the CUNY Start instructional approach in mathematics:

1. The use of a highly detailed curricular document as a primary resource for instructors;

2. An emphasis on number relationships, which serve as the instructional starting point (rather than mathematical rules and procedures);

3. A pedagogical approach that elicits student talk and discussion through questioning; and

4. Explicit attention to students' organizational and study skills.

In this section, we offer examples to show how these features are implemented and highlight student and instructor perspectives on these dimensions of CUNY Start instruction. We use data collected from developmental mathematics courses at the four evaluation college sites to draw comparisons between CUNY Start and traditional developmental mathematics offerings.

4.1 The Curricular Document

The first feature of the CUNY Start mathematics instructional approach is the highly detailed copyrighted curricular document (see Appendix B for excerpts from this

document), which spans upwards of 700 pages and includes 47 daily 2.5 to 3-hour lessons to be delivered over approximately 12 weeks. Each lesson includes detailed instructional notes that explain both the rationale for the activities and a guide to facilitation. There is no textbook; instead the curriculum includes student handouts for in-class activities and homework, which students organize in a binder along with their notes. The curriculum includes four summative assessments.

The following is an excerpt from the instructional notes for one activity found in the curricular document. The activity is called "Counting Cubes" and is one portion of the seventh class of the semester (see more details on this lesson, including the student handout, in Appendix B.2):

> Rather than starting by defining the meaning of an exponent, this functions activity will give your students a chance to reveal their prior knowledge if they have it. It also can be used to help them understand (for many, for the first time) why we can say "five squared" in place of "five to the second power" and "two cubed" in place of "two to the third power."

> You will need a set of 1-inch foam cubes and a single ruler for this activity. Before giving out the handout, take 36 one-inch cubes and arrange the three figures on a central table. It is best to label them "figure 1", "figure 2", and "figure 3" using index cards in order to correspond to the handout.

> Have your students come around and look at what you have done. Ask them to describe what they see. You will likely hear "cube" and "square" among the replies. Put these words on the board as they come up, and try to have students clarify what they mean. Elicit clarity that these figures are not "squares"—squares are flat and only exist in flat surfaces.

> Ask students for their observations about a cube. Each cube has six square "faces." You may introduce the word "edge" to describe the segments that connect the faces. In trying to describe the pattern, students may say "one, two, three." If this happens, question them until they make and agree on more precise statements. It would be good to have a ruler there so that they can make some statements about length.

And while in theory it might be nice to discuss area and volume, it is probably too time-consuming at this moment, and it could lead students to try to use formulas to solve the upcoming problem rather than exploring, coming up with their own approaches, and collaboratively making sense of it in their own ways. (CUNY Start mathematics curriculum excerpt)

This excerpt is characteristic of the curricular document in that it provides specific guidance on how the lesson is designed to unfold and how to facilitate student participation. It includes likely student responses as well as suggestions on how to follow up on those responses. Throughout the curricular document, lessons are explicitly situated within the broader scope and sequence of the program, with references to previous and upcoming content.

The CUNY Start curriculum presents topics in a carefully considered sequence, with key topics, including those on number relationships, algebraic expressions, and functions, fully integrated and revisited throughout the document. Hinds (2009) explains the rationale for this approach: "Mixing content helps students to make important connections between topics. . . . As soon as a number topic is considered, we may incorporate it into our work with expressions and functions to increase the challenge, or we may use it as a basis for introducing a new algebraic idea" (p. 23). In the first two weeks of CUNY Start mathematics, students engage in topics one might expect to see in the early weeks of a developmental arithmetic course (i.e., signed numbers, decimals, and exponents), as well as what may be seen as more advanced algebraic topics like function relationships and understanding and manipulating expressions and equations. For example, Table 1 presents the learning objectives for the first class of the semester (see Appendix B.1 for instructional notes and handouts for this class):

Table 1
CUNY Start Mathematics Learning Objectives for Class 1

Topic	Learning Objective
Arithmetic/pre-algebra	Students are guided to think about decimal tenths and hundredths as numbers of dimes and pennies. Students draw on this place value understanding to compare and add decimal numbers.
	Drawing on what they already know about multiplication, students describe whole-number multiplication in terms of "groups" and represent groups as repeated addition. Combining this with the conceptual work they did with decimals, students multiply a whole number by a decimal.
	Students interpret positive numbers as "money held" and negative numbers as "money owed," and use these interpretations to add numbers.
Functions/algebra	Students build a table of input-output values for a linear relationship presented in a meaningful context and generalize a function rule for the relationship.
Organizational/ study skills	Students learn and begin to use the 3-ring binder routine for organizing their class materials.

As noted in Scrivener et al. (2018), instructors largely follow the CUNY Start mathematics curriculum with fidelity. Because of the high level of detail included in the curricular document, instructors have fewer decisions to make about how to organize their class time and are thus able to home in on making specific instructional decisions while teaching (e.g., how to respond to a wrong answer). Notably, the CUNY Start training and professional development model focuses on supporting instructors to manage those in-the-moment instructional decisions.

These two curricular features—the integration of number, algebraic expressions, and function topics and the detailed instructional guidance—contrast with the curricular approach of most traditional developmental mathematics courses. A typical developmental mathematics sequence often comprises at least two courses. In one common configuration, which is offered at three of the four colleges in this study, the first course focuses largely on number topics and the second one focuses largely on expressions and functions. Faculty in mathematics departments often work together to select a common textbook and other instructional resources (e.g., a software package) that meet the learning objectives of the developmental mathematics courses. Typically,
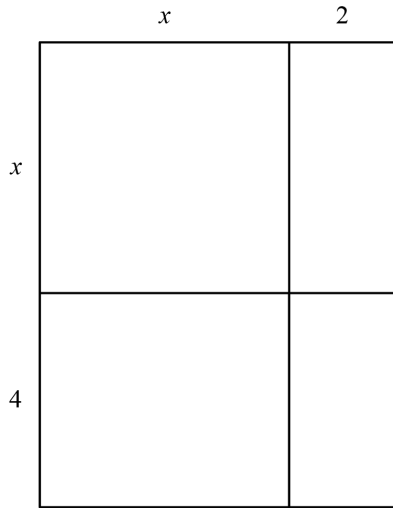
these materials include limited instructional guidance, and faculty have discretion on how to implement the curriculum. In some departments, instructors may also supplement or replace departmental materials with those of their own choosing.

The CUNY Start mathematics curriculum, including instructional guidance, is edited and refined regularly by CUNY Start professional development coordinators in response to their classroom observations and feedback from instructors. By contrast, under the traditional model, after having selected a textbook and resources as described just above, any refinements, including decisions about how to introduce a concept, which problems to write on the board, and which questions to ask students during class, are made by faculty on their own. As we discuss below, these curricular and programmatic differences have significant implications for instruction.

4.2 An Emphasis on Number Relationships

The second feature of the CUNY Start instructional approach is that mathematical rules and procedures are never the instructional starting point for a topic or concept. Instead, the CUNY Start mathematics learning goals focus on several interconnected elements of mathematical proficiency, including conceptual understanding, reasoning, and fluency with procedures (Kilpatrick, Swafford, & Findell, 2001). Hinds (2009) provides justification for this feature of the curricular design: "It [means] students who forget a rule may not be helpless—they can think about the mathematical relationships and may be able to work their way back to a solution. . . . For students who successfully memorized some of the mathematics rules in an earlier class, this adds important justification and depth to their understanding" (p. 28). Lessons often begin by asking students to explore number relationships, a visual model, or a real-world situation.

For example, in a lesson on multiplication of binomials, students are first introduced to a figure of four adjoining rectangles for which they are asked to find an expression that represents the area:

After students work on this task, the class discusses two possible expressions. The first expression shows the product of the sides of the large rectangle; the second shows the four smaller areas combined:

$$(x+4)(x+2) \quad \text{and} \quad x^2 + 2x + 4x + 8$$

Using the rectangles and the expressions, the instructor then guides the students through a series of questions (e.g., "Is there a relationship between these expressions? Are they equal? Why?"). In the discussion, students are expected to recognize that the two expressions must be equal because both represent the area of the large rectangle. The instructors also connect the expressions to the previously learned distributive property. Only after all of this work do instructors make reference to the "first, outer, inner, last" (FOIL) process:

> By this point, one or more students will likely have made the connection between this process and "FOIL"-ing. Whether they did or not, it makes sense to review the meaning of this acronym (because professors or other students may refer to it) and connect it to the distributing that we just did. Make sure that students see that this process gives us four terms, which are the same as the individual (smaller) areas in the figure. (CUNY Start mathematics curriculum excerpt)

Thus, students are introduced to FOIL, but only after exploring a visual model for the multiplication of binomials.

Many lessons feature real-world contexts, which are intended to help students see and understand number relationships and increase student interest and engagement. For example, functions are first introduced in the context of earning a base wage plus commissions: "You take a job at Best Buy selling smartphones. Your base pay is $150 per week. For each smartphone that you sell, you earn an additional $18." (See Appendix B.1 for more information.) A later lesson extends this understanding to functions using algebraic forms. Similarly, signed numbers are introduced in terms of money held and debts before students are asked to complete decontextualized computation problems.

CUNY Start instructors described the limitations of other approaches to teaching mathematics in which formulas or procedures are presented first as the primary instructional focus. For example: "The method they used [in high school] is: 'Quote a formula, plug in numbers, done.' It's quick, but there is no meaning and nothing that would help you check your answer." Another instructor elaborated:

> The goal of our program is not to introduce formulas, not to introduce rules, just have them come up with the relationships. I mean I've heard the students [say], "*a* squared plus *b* squared is *c* squared." Okay, what does that mean? They don't know. (CUNY Start instructor interview)

When asked about their experience in their CUNY Start mathematics classes, students talked frequently about the prevalence of real-world contexts in the curriculum.

> The teacher will put it as, like, money. The teacher puts it more as like realistic, real-life problems. In high school they just taught it to us quick and fast, and it was a little bit difficult to understand. (CUNY Start student interview)

For these CUNY Start students, the use of real-world contexts to introduce number relationships made the content more relatable, approachable, and easier to understand.

Classroom observations of traditional developmental mathematics courses largely revealed an instructional approach in which rules were presented first, followed by student practice and then, in some cases, by applications to real-world contexts. For

example, in an arithmetic class, the instructor began by showing students how to calculate the mean before asking students to complete the practice problems with decontextualized numbers. Finally, students were asked to complete a series of contextualized problems (e.g., finding the average wage at a company given a list of wages). One instructor of traditional developmental mathematics courses described their instructional approach this way: "I teach first. I'm going to lecture first, then give examples [from the practice problems], and then ask students to do similar practice problems in class."

Seventy-seven percent of surveyed traditional developmental mathematics instructors reported that, when introducing a new concept, they often or always "show the class how to do something before having students do it on their own," compared to 13 percent of surveyed CUNY Start mathematics instructors. Surveyed CUNY Start and traditional developmental mathematics instructors reported asking students to work on real-world problems at roughly similar rates: about 54 percent of both groups said they "often or always apply mathematical concepts to real-world problems." However, our data show that real-world problems were used differently in CUNY Start, often as the starting point for an exploration of number relationships rather than as an application problem to practice using a learned procedure.

4.3. Eliciting Student Talk and Discussion Through Questioning

The third feature of CUNY Start mathematics instruction is the expectation that students explain their thinking, justify their answers, and discuss their approaches with their classmates. Analysis of the use of class time in observed classes shows that CUNY Start instructors lectured for about 2 percent of the total time observed, which suggests that students are expected to contribute a great deal in class. In whole-class discussions, students are asked to explain their thinking, sometimes as the instructor writes at the board in response to student suggestions. Often this occurs after students have written their own solutions to problems on the board. In 11 of 12 CUNY Start mathematics classroom observations, multiple students solved problems on the board and were then asked to explain their work. This practice of showcasing student work gives students opportunities to practice explaining their thinking, while also demonstrating multiple solution approaches for a given problem (Shimizu, 1999). Emphasizing multiple solution

approaches is thought to enhance critical and flexible thinking in mathematics (Levav-Waynberg & Leikin, 2012), and this strategy complements the instructional focus on number relationships.

Student discussion in a CUNY Start mathematics class is mainly initiated through instructor questions. Suggested questions appear in the instructional notes in the curricular guide, but crafting the right kind of in-the-moment questions during class discussions is an important element of the training and ongoing professional development CUNY Start instructors receive. A CUNY Start professional development coordinator explained that CUNY Start instructors are trained to avoid "leading questions that provide 80 percent of the language of the thinking for the students" and instead to ask questions that invite student thinking (CUNY Start staff interview). Examples that appear in our classroom observations data include: "What do you see? What are we looking at here? How did you get that answer? Do you agree with [your classmate's] response? Why or why not?" Instructional notes for one lesson explain the use of this type of questioning: *"A pedagogical goal here is not to see how helpful we can be, but to see how little help we can give while students reach this conclusion themselves"* (CUNY Start mathematics curriculum excerpt).

When asked about their experience in CUNY Start, students frequently discussed the expectations around student participation and the use of questioning:

> In CUNY Start . . . you have to participate. And that's what
> I needed to really learn. Because when you are sitting down
> and just looking at the board, you are not really processing
> it. (CUNY Start student interview)

> He doesn't just give answers. If we ask him, "Is this
> correct?," he won't tell us. He'll just say, "I don't know.
> You figure it out," which is pretty unique because he wants
> us to learn on our own. Basically he's saying, "I'm not
> going to be there with you on the day of the test." And he
> knows we can do it. (CUNY Start student interview)

In interviews, many students reported benefits to this approach, including enhanced engagement, learning, and confidence. However, CUNY Start instructors do encounter challenges in implementing this component of the instructional approach. They

reported in interviews that learning to craft open-ended questions that facilitate student learning is the most difficult aspect of learning to teach in CUNY Start:

> One mistake I was prone to making was being a little too leading because I wanted to help the students, and so you are like throwing little hints. So instead of asking a student, "Okay, so what are we looking at and what we can do?," I would be a little bit more direct: "How could you factor that?" You don't want to be that direct. You want them to come up with the idea that they need to factor on their own. (CUNY Start instructor interview)

These open-ended, non-leading questions are intended to allow for productive struggle, in which "students expend effort to make sense of mathematics, to figure something out that is not immediately apparent" (Hiebert & Grouws, 2007, p. 387). Although this kind of struggle is intended to be productive (i.e., students are struggling with problems where the solution is "within reach"), it can cause discomfort for both students and instructors. In particular, CUNY Start instructors discussed a learning curve for students at the beginning of the semester: "It's hard because they are just used to a traditional environment. They are used to spectating also in a sense. And our program requires many to really struggle to do the work, and we don't just give it to you." One instructor elaborated on the nuances of implementing this approach:

> How long should a student struggle? That's a question I don't think anyone could answer because the circumstances are different for every student. I could have a student who is ready to go in tears after one minute and this [other] student will freely struggle with me for 15 minutes and love every minute of it. (CUNY Start instructor interview)

Because of these challenges, ongoing professional development for CUNY Start instructors focuses extensively on crafting good in-the-moment questions, responding to student misunderstandings via questioning, and setting the conditions for productive struggle.

In the observed traditional developmental mathematics courses, instructors asked questions of their students, but there were few instances of the kind of facilitated student discussion described above. In the survey, 81 percent of non-CUNY Start developmental mathematics instructors reported they often or always "pose questions seeking

information from students." In the observed courses, the aim of those questions was typically to elicit a correct answer. Sample questions observed in traditional developmental courses include: "How did you simplify? What did you get for this problem? Can I add these two [terms]? How do you write [this decimal] as a fraction?" If students could not provide the answer, instructors were likely to provide it. A developmental mathematics faculty member explained that they do elicit ideas and responses from students, but when students are struggling, "that's when I move on and start to give them the solutions" (developmental instructor interview).

According to observation data, developmental mathematics instructors spend 18 percent of class time facilitating discussion among students, compared to 49 percent of class time for CUNY Start instructors. Survey results show that 97 percent of CUNY Start instructors "often or always ask students to explain out loud how they solved a problem," compared to 64 percent of developmental instructors. Some developmental instructors reported that time constraints are an obstacle to more interactive approaches. One explained, "Think about it. Thirty-five chapters to go over in 12 weeks is a lot. And each chapter takes a lot to bring the lesson out. So that's why I ask them to come to my office hours" (developmental instructor interview). Expanded time for each class session is among many structural features that contribute to the ability of CUNY Start instructors to successfully implement the curriculum. These programmatic elements are discussed in Section 5 of this paper.

4.4 Building Students' Organizational and Study Skills

The fourth feature of the CUNY Start mathematics instructional approach is a strong focus on building organizational and study skills. This is built into the class requirements and is explicitly addressed in the instructional notes throughout the curricular document. Students are responsible for maintaining a binder of their work, including all handouts, notes, assessments, and homework assignments (there is no textbook or software). The binder is checked for completeness and organization by the instructor in advance of every exam.

This practice helps to foster students' organizational skills, but more critically, it provides students with an extensive resource. The binder includes a vocabulary index that students are responsible for populating throughout the semester with new words, along

with examples and descriptions, as well as the date and/or handout on which the word appears. The instructional notes also provide guidance on helping students record their thinking on handouts that will be placed in their binder. For example, these notes accompany an activity in which students solve number puzzles:

> When you see a student who is methodically writing down pairs of numbers that they are testing, praise that in front of the other students. More typically, a student will write down a pair and then erase it when it does not solve the puzzle. Encourage students to keep a list and cross out the pairs that do not work, both to keep track of what has already been tried, and also because later on it gives you and the student important information on what was done. Math handouts covered in student work may not be attractive from a student's point of view, but they are very valuable to the teacher who is curious about understanding student thinking and to the student who may later forget what they did and thought. (CUNY Start mathematics curriculum excerpt)

One way students are expected to use their binder is in preparation for assessments. The program includes four major assessments, and the instructional notes include specific guidance on preparing for them. Before the first assessment, instructors facilitate a discussion about how to study for a mathematics test. After soliciting student ideas, they provide a list of recommendations, the first of which is: "Studying for a math test must include doing math problems. Looking at notes is not enough." In preparing for assessments, students are instructed to look in their binders for problems they found challenging: "Copy any problems that challenged you. . . . After doing these problems, check your work against the solutions in your binder" (CUNY Start mathematics curriculum excerpt). The curriculum also recommends that students create their own problems that look similar to the problems they found challenging. Class activities also include three timed quizzes designed to acclimate students to a timed test-taking environment they will likely encounter in future college coursework and help students employ strategies like skipping difficult questions at first and returning to them later.

In traditional developmental mathematics classes, students are expected to take notes. In the instructor survey, the majority of both traditional developmental (77 percent) and CUNY Start instructors (83 percent) reported that they often or always ask students

to "use notes when working on activities in class." While individual developmental instructors may provide guidance around taking notes and preparing for assessments, a review of the sample developmental mathematics syllabi collected from participating colleges showed that only one (from a quantitative reasoning course) included note-taking as part of the course grade. Several of these sample syllabi showed that homework is completed via software. This suggests that many developmental mathematics students do not have ready access to a collection of problems with information on how they completed them. The lack of a textbook and software in CUNY Start gives students more responsibility for curating course materials. For example, textbooks may provide a glossary of important terms, but CUNY Start students must create their own glossary.

4.5 Summary of Instructional Features

Taken together, these four features result in a mathematics instructional experience that is unlike most postsecondary mathematics courses. Specifically, students are asked to discuss number relationships (both in contextualized and decontextualized scenarios). Through those discussions, they may observe multiple solution pathways. Those pathways are sometimes compared and refined to generate an efficient rule or approach to problem solving, but this is the endpoint of the conversation, not the start. This kind of learning environment is facilitated by an instructional method that hinges on the strategic use of open-ended questions. An extensive curriculum guide provides instructors highly detailed guidance, so they can focus on nuances in instruction. The CUNY Start program positions students as thinkers and doers of mathematics. They are expected to talk about mathematical relationships, strategize on ways to approach problems, and provide justification for their approach. Expectations around curating and organizing their work are an additional way students are expected to actively contribute to their learning. As documented above, these features are very different from what we observed in traditional developmental mathematics courses. In the section that follows we describe several components of the CUNY Start program that facilitate this dramatically different instructional approach.

5. Programmatic Elements That Support Implementation

The implementation of these four features is supported by a number of broader program components. Our data show the ways in which these program elements enable CUNY Start instructors to implement the program's unique approach to teaching mathematics. As mentioned earlier, these program elements include the instructional intensity, the staffing model and extensive professional learning resources for instructors, and integrated academic and nonacademic student supports. (For a more detailed description of the broader CUNY Start program, see Scrivener et al., 2018).

Instructional intensity. The CUNY Start mathematics class meets for about 11 hours per week for a semester, typically 2.5 or 3 hours per day for four or five days each week. The program is completed in either one phase (for students who pass an exam at the end of Phase 1) or two phases. By contrast, a typical developmental arithmetic and elementary algebra sequence stretches over two semesters, typically with 3 to 6 hours of instructional time each week. Although the total number of instructional hours may be similar, daily instructional blocks of over two hours allow for the kind of hands-on student-led exploration of number relationships and multiple solution pathways illustrated throughout the sections above. CUNY Start students who do not successfully pass Phase 1 are referred to Phase 2, where they will have a second opportunity to pass the course for several additional weeks of instruction before the next semester begins.

Training and professional development. Another key program component is the staffing model, coupled with the training and professional development provided to CUNY Start mathematics instructors. In contrast to the pool of CUNY faculty, which includes significant numbers of adjunct instructors, CUNY Start instructors are full-time employees. When hired, CUNY Start instructors participate in a paid semester-long 25-hour per week apprenticeship in which they are assigned to the classroom of an experienced CUNY Start instructor. At the beginning of the semester, apprentices in mathematics largely observe, and as the term progresses, they are given the opportunity to lead portions of class. CUNY Start also employs several professional development coordinators tasked with providing training and support to all instructors and revising the curriculum. During the apprenticeship semester, new instructors have weekly meetings to preview upcoming lessons, reflect on lessons observed, and receive training on the

instructional approach. Given how different the CUNY Start instructional approach is compared to mathematics teaching in most U.S. educational contexts, instructors reported how important the apprenticeship semester is:

> CUNY Start's pedagogy is totally different. Especially if you come in with a teaching identity, you need [the apprenticeship]. This needs to be sort of spoon fed to you. Because if you drop me in the classroom with this [curriculum], I'm going to run for the hills. So I think it's crucial. I don't see how I could have done it any other way. I had to see a teacher in action. I had to see the student talk. (CUNY Start instructor interview)

Even after CUNY Start instructors have completed the apprenticeship and take on their own class, they are observed by a professional development coordinator. An instructor described the nature of those observations:

> They'll come in and observe maybe once or twice a semester. There's always a wrap-up debriefing meeting afterwards [focused on] analyzing a piece of like a transcript of the class. So [the professional developer] does a lot of writing down, recording what goes on during the class, and then they'll pull out specific interactions and ask about why we asked a certain question, what we were thinking about. (CUNY Start instructor interview)

These observations are more frequent for new instructors, but even experienced CUNY Start instructors are observed regularly for the purposes of receiving feedback on their implementation of the instructional approach. Both instructors and professional development coordinators emphasized how these observations and meetings are important to maintaining and improving the quality of instruction. A forthcoming paper from the evaluation will explore CUNY Start's approach to training and professional development in greater depth.

Student supports. Embedded academic and nonacademic student supports also bolster CUNY Start's challenging instructional approach. For academic support, the program uses tutors, program graduates who have received training on the instructional approach, for in-class and out-of-class tutoring. In-class tutors attend class and assist students as they work individually and in groups. In addition, some CUNY Start

mathematics classes are assigned a support instructor who is present in the classroom part-time. Support instructors have completed an apprenticeship and are available to lead some lessons as well as provide support to individuals and groups. In interviews, instructors talked about the value of having, in the words of one interviewee, "an extra set of eyes" in the classroom to see when and how students are struggling.

The program's nonacademic supports are delivered largely by program advisors who teach a weekly college success seminar and meet individually with students as needed if challenges arise. Each CUNY Start mathematics class is assigned an advisor who meets weekly with instructors to discuss student progress and behavior. This provides an opportunity for content area instructors to raise issues such as absences, tardiness, disengagement, or challenges in the classroom dynamic. Advisors and instructors can strategize on how best to address those concerns. Advisors have a case load of 75 students, which allows for individual student attention not afforded to most advisors in the traditional college setting. CUNY Start mathematics instructors talked about how they work closely with their students' advisors to intervene and support students. One instructor described how they have addressed negative student behaviors in the classroom, like disengagement and use of cell phones:

> So my advisor might know what I'm talking about because they have seen it, and we share stories and situations. So we usually know, we start talking to each other, "Okay, it's time to talk to the student." So we have done that plenty of times. We sit down with the student [together]; we discuss what's going on. (CUNY Start instructor interview)

Advisors talked about the particular role they play in helping students acclimate to the CUNY Start mathematics program, which includes explicitly addressing the expectations around discussion, justifying their solutions, and responding to instructor questions. The supports provided by advisors, professional development coordinators, and the tutors and support instructors are critical to both student and instructor success in CUNY Start mathematics.

6. Is CUNY Start Working?

The ongoing IES-funded evaluation of CUNY Start (of which this paper is part) is providing the first experimental evidence of the program's positive short-term effects, which appear to be particularly strong in mathematics. (Longer-term effects will be tracked and reported at a later date.) Scrivener et al. (2018) found substantial differences in the proportion of CUNY Start students who were college-ready in mathematics after the program semester compared to control group students[10] (57 percent versus 25 percent). College readiness in mathematics in this analysis was measured using student performance on the CUNY Elementary Algebra Final Exam. These results are noteworthy given that more than half of the CUNY Start students in the study would have needed two traditional semesters of developmental mathematics based on their placement scores. The analysis also found that CUNY Start students earned fewer college credits than the control group of students in the program semester. This result is expected given that CUNY Start students defer matriculation by one semester and thus are not able to enroll in college-level courses.

These early results from the rigorous evaluation build on previous quasi-experimental and descriptive studies of CUNY Start. Using a propensity score matching (PSM) approach to create comparison groups of students that are similar based on observed characteristics, a study conducted by CUNY research and evaluation staff found that 53 percent of CUNY Start mathematics students reached college-level proficiency in mathematics;[11] meanwhile, only 10 percent of similar students referred to traditional remediation in mathematics who enrolled directly in a CUNY college reached college-level proficiency after one semester (Allen & Horenstein, 2013). The study also considered outcomes in reading and writing, in addition to mathematics. Positive effects of a smaller magnitude were found for achieving proficiency in reading (24 percentage points) and in writing (36 percentage points). The study also found that the subset of

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[10] Control group students had access to the standard courses and services at the colleges, including standard developmental education.

[11] At the time of the study, students referred to developmental mathematics in CUNY were deemed college-ready if they attained a minimum passing score on the CUNY Elementary Algebra Final Exam (CEAFE) in addition to earning a passing grade in their developmental mathematics class.

CUNY Start students who went on to matriculate in a CUNY college attempted and earned more credits and had higher rates of persistence than the comparison group.

Another internal study using PSM and focused on the effects of CUNY Start on academic momentum found that CUNY Start students were more likely than similar non-participants to take and pass gateway courses in their first year, earning similar grades in English gateway courses and slightly lower grades in mathematics gateway courses (Webber, 2018). This same study also found that Black and Hispanic CUNY Start students experienced larger gains in gateway English and mathematics completion rates than Asian and White CUNY Start students. In another study, using PSM and a discrete-time hazard approach, Allen (2015) examined the relative effects of some alternative developmental education pathways[12] in CUNY and found that CUNY Start was associated with more positive outcomes in terms of completing developmental education requirements, passing introductory-level college mathematics and English courses, and earning a degree.

7. Conclusion and Implications

CUNY Start was created as community colleges broadly were beginning to understand the challenges associated with traditional developmental education. Perhaps because of the program's origins in HSE programming and its attempt to address the large proportion of HSE completers placing into developmental mathematics due at least in part to low mathematical proficiency, the CUNY Start approach to mathematics is distinct from many other developmental mathematics reforms. Corequisite remediation, for example, seeks to ensure that students enroll directly in gateway mathematics courses, bypassing stand-alone developmental education, and configures supports designed to increase the probability of success in those courses. CUNY Start likewise aims to increase the likelihood of students' success in college mathematics, but it does so through a prior semester of intensive instruction designed to improve mathematics proficiency and students' capacity as learners. As another example, mathematics pathways reforms

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[12] The alternative developmental education pathways that Allen (2015) studied were University Skills Immersion Program (USIP), CUNY Language Immersion Program (CLIP), and CUNY Start.

aim to accelerate completion of developmental requirements and align mathematics course-taking to students' programs of study. CUNY Start also seeks to accelerate completion of developmental mathematics requirements but does so by compressing content typically spread over two semesters into one. Relative to most developmental mathematics reforms, CUNY Start attends to curriculum and pedagogy more explicitly and has developed the program's key features to reflect this focus.

Four key features of the CUNY Start mathematical instructional approach are (1) the use of a highly detailed curricular document as a primary resource for instructors; (2) an emphasis on real-world contexts and number relationships, which serve as the instructional starting point (rather than rules and procedures); (3) a pedagogical approach that elicits student talk and discussion through questioning; and (4) explicit attention to students' organizational and study skills. These features differentiate CUNY Start from traditional developmental mathematics in a number of ways. Unlike in CUNY Start, curriculum in traditional developmental mathematics is not prescribed and teaches concepts in a sequential order seemingly based on difficulty. Moreover, it rarely integrates instructional guidance that helps instructors anticipate and address students' misunderstandings, one of the key program features of CUNY Start. Most traditional developmental mathematics lessons first teach students rules and procedures, followed by opportunities for practice and application. CUNY Start lessons frequently begin by asking students to explore number relationships and bring in rules and formulas near the end. In traditional developmental mathematics education, instructors typically provide direct responses to students' questions, stating whether the student is right or wrong and limiting the productive struggle that CUNY Start professional development coordinators encourage in service of deeper understanding. Lastly, traditional developmental mathematics courses rely on texts and instructional software that limit students' ability to document their own learning and assemble a resource to aid their studying.

The individuals who developed CUNY Start built out structures, processes, and resources specifically designed to increase the likelihood that the instructors could successfully implement this instructional approach to mathematics. CUNY Start leadership and professional development coordinators were intentional about the structure of the program and embedded robust supports for instructional staff. The

program schedule provides sufficient time for instructors to deliver the lessons as envisioned. Newly hired instructors apprentice in the classrooms of experienced instructors, witnessing firsthand how the curriculum is enacted and assuming increasing teaching responsibilities over the course of their first semester. Continuing classroom observation generates formative, actionable feedback for instructors and continuous refinement to the curriculum. Mathematics tutors provide in- and out-of-class assistance to students, and advisors collaborate with instructors to confront challenges that may arise and hamper student performance.

This paper is part of an evaluation that finds CUNY Start has significant positive effects on students achieving college readiness in mathematics. The proportion of CUNY Start students who were college-ready in mathematics after the program semester was more than twice that of the control group. More than half of these CUNY Start students would have been assigned to two semesters of remediation based on their placement exam scores. While it is difficult to know what features of CUNY Start drive that difference, it is notable that the program effects are much stronger in mathematics than in reading and writing (Scrivener et al., 2018). It is possible that the focus on curriculum and instruction, which showed stronger contrast with traditional developmental coursework in mathematics than in reading and writing, contributes to these outcomes.

Perhaps the most striking contribution of CUNY Start harkens back to its development. It was developed based on the idea that it is within the power of colleges to dramatically change the way mathematics, reading, and writing is taught in order to increase the mathematical and literacy proficiency of students who enroll in college with significant academic (and nonacademic) weaknesses. The evidence uncovered in this evaluation strongly suggests that CUNY Start's structures, processes, and resources enable instructors to teach mathematics in a different way that may boost student achievement. For practitioners seeking a road map to improve teaching and learning in mathematics, CUNY Start provides the field with a clear model for improving academic outcomes and building mathematical proficiency.

Appendix A: Data and Methods

Data for this paper were collected within the context of a random assignment evaluation of CUNY Start (see detailed description of the evaluation methods in Scrivener et al., 2018). This evaluation was conducted in 2015 and 2016 at four CUNY community colleges that have CUNY Start programs. Some 3,835 CUNY Start-eligible students were randomly assigned to CUNY Start or to receive regular college programs and services, including traditional developmental coursework.

For this paper's analysis, a subset of data sources was analyzed, presented in Table 2. Specifically, this paper draws on a review of curricular materials as well as interviews with key stakeholders, a survey of instructors, and classroom observations. Data were collected from all four colleges, except where noted.

Table A.1
Summary of Data Sources

	CUNY Start	Developmental Mathematics
Mathematics curricular materials	full curriculum	7 syllabi
Interviews and focus groups:		
Instructors/faculty	10	10
Professional development coordinators	2	N/A
Students	47	27
Instructor/faculty survey respondents	34	152
Classroom observations	12	9

The CUNY Start mathematics curricular guide, which includes instructional notes, student handouts, and assessments was reviewed in detail. Syllabi for seven traditional developmental math courses offered at three of the four colleges were also reviewed. These syllabi included course objectives, course texts, assignments, and grading criteria. Interviews and focus groups were conducted with CUNY Start math instructors, CUNY Start professional development coordinators, and developmental math faculty. Students enrolled in both CUNY Start and in traditional developmental mathematics courses were also interviewed in individual interviews and focus groups. Developmental math faculty and CUNY Start instructors were surveyed (see Scrivener et al., 2018, for details on survey administration and response rates). Finally, this analysis draws on observations of CUNY Start mathematics classrooms and developmental math

classrooms. Two observers visited each class: One took fieldnotes and the second recorded categories of instructional activities at five-minute intervals.

The CUNY Start mathematics curricular materials were closely analyzed for this paper. Researchers uploaded classroom observation fieldnotes and interview and focus group transcripts to Dedoose for data management and coding. Drawing on the coding scheme developed for the larger evaluation, this analysis focused on a selected set of codes relevant to math instruction, including "curriculum," "pedagogy," "student learning," and "professional development," among others. The classroom observation time-series (i.e., recording of instructional activities at five-minute intervals) was analyzed descriptively to understand the prevalence of instructional approaches in observed classes. Observation, interview, and survey data were triangulated and contextualized within a close analysis of the CUNY Start curricular materials to affirm themes presented in this paper.

Appendix B: Curricular Excerpts (spring 2019 revised version)

B.1 Class 1 Lesson Excerpts

B.1.1 Class 1 Instructional Notes

- **Number Puzzles**

These Number Puzzles were adapted from *College Preparatory Mathematics*. Eventually, we will use these types of puzzles so that students may practice combining, multiplying, and factoring expressions. Even when limited to numbers, though, these are an engaging way that students may practice factoring numbers and do decimal and signed-number arithmetic.

This is a good warm-up for your students as they enter the room on the very first day of class. Rather than sitting around in awkward silence waiting for the bulk of the students to arrive, students can get started doing math right away, which sets a good tone. With written directions, they hopefully will not need much attention from you in order to get started.

However, do not wait very long before interrupting students' work on the number puzzles to have an all-class discussion. Assure them that they will have more time to work on the puzzles after this discussion. When some students have reached the puzzles with decimals, begin an all-class discussion by focusing on the sample puzzle at the top and asking if anyone can explain how this puzzle works. When a student describes multiplying 3 and 5 for the top circle, ask how they know to multiply. They can refer to the instructions or to the symbol in the puzzle. Regardless, be sure to discuss the word "product". Do the same thing when a student explains the bottom circle, discussing "sum" in the process. Then discuss the puzzle with 18 and 3 to make sure that students understand how these puzzles work, even when the base numbers are not both given.

Most of our students are weak in decimal understanding and operations. After going over those two puzzles, write **$2.73** on the board and ask students to tell you what they see.

Ask students if they can create this amount of money using only dollars, dimes, and pennies. They will most likely tell you they used 2 dollars, 7 dimes and 3 pennies. Push for other possibilities as well. They could come up with many solutions, including the examples below. For international students, you may need to discuss the value of each of these coins.

$2.73

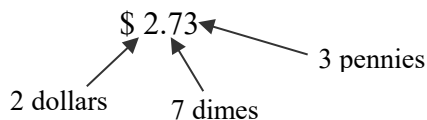| 2 dollars	0 dollars	1 dollar
7 dimes	0 dimes	17 dimes
3 pennies	273 pennies	3 pennies

Direct students to begin taking notes on a separate sheet of notebook paper at this point. Some students will not have notebook paper with them. Bring some extra sheets to share with them, or gently ask fellow students to share a few sheets with their classmates this time.

Point at each of the digits in the original quantity, and with the following questions, students should help you to label the place values. See the figure below.

This 2 is the number of what?
This 7 is the number of what?
This 3 is the number of what?

$ 2.73

3 pennies

2 dollars   7 dimes

For many students, it is helpful to think of the tenths place as the number of dimes, and the hundredths place as the number of pennies. Using the terms "tenths" and "hundredths" is not important or necessary here. Teachers have lectured students many times over the years about the names of these places, without successfully focusing student attention on their relative sizes.

Follow this up by asking about another example or two of money written as a decimal number, such as $7.16 and/or $3.09, to check if this dimes/pennies idea works for more than just $2.73.

Once students imagine decimal places as dimes and pennies, gauge their understanding with the following questions. These comparisons confound students with little conceptual understanding of decimals, but when thought of as the number of dimes and pennies, they become much easier.
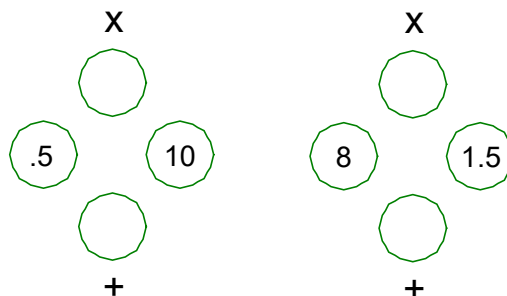
Which is larger, .4 or .08? Why?

Which is larger, .6 or .60? Why?

The second of these comparisons can be a real eye-opener for students. Many have been told without any meaningful justification that they can add a zero to .6 and it means the same thing. Students may have believed their teachers, but belief should not be mistaken for understanding. With the dimes/pennies formulation, .6 = .60 will make sense to some students for the first time.

Reinforce, reinforce, reinforce. As you encounter decimal numbers in this and other activities, whether one or two decimal places are shown, point to a digit and ask *"This the number of what?"*
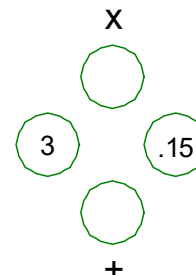
After this discussion of money and decimal places, students may have enough understanding to calculate the sums in the puzzles with decimals, such as the examples to the right. The thing that might trip them up is the meaning of 10 and 8. Because no decimal point is visible, they may feel uncertain about how they can think of them in terms of money.

X   X

.5   10   8   1.5
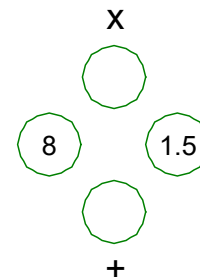
+   +

30

Ask students if they have seen and understand each of the following prices that might appear in a store. This will quickly remind them based on their own experience that a number written without a decimal point can be imagined as a number of dollars and that the decimal point is "hiding" to the right when it is not shown.

Handbags $5

Handbags $5.00

This discussion should ensure that students are capable of determining the sums without a calculator. You might draw a new example or two on the board to assess this. At this stage, focus only on the sum.

X

(3)   (.15)

+

In order to determine the products for the decimal puzzles, students can of course use the traditional algorithm for multiplying decimals. The trouble is that many students struggle with these procedures (especially placement of the decimal point) and have little or no ability to judge the reasonableness of their answers. This is an ideal place to assist students in deepening their understanding of multiplication so that they have alternatives to the standard algorithm and so they have a "reasonableness check" for any method they choose to use. This discussion of multiplication is a critical one because related ideas will appear in several sections of this course. It cannot be skipped.

X

(8)   (1.5)

+

Ask your students to do the following multiplication: 3×5

They will be able to tell you the product is 15, but then we must ask them — "***Why is it 15?***"

Our goal in this discussion is to relate multiplication to adding groups. Once a student describes the multiplication in this way, including what we should do with the three 5's (what operation we should use), have them add it to their notes:

$$3 \times 5$$
$$\text{Three groups of } 5 = 5+5+5 = 15$$

It is important for students to also articulate that 3×5 may also be interpreted as "*five groups of 3*" or $3+3+3+3+3$. Have students record this in their notes too.

Students can now combine two ideas developed in this activity to determine many products involving decimals — translating decimals into money and the idea of multiplication as groups. Discuss the last puzzle on the front of the handout, asking a student to explain it using these ideas.

$$8 \times 1.5$$
$$\text{Eight groups of } \$1.50$$

Students now have an alternative to the traditional algorithm. Most students can add these amounts accurately.

$$\$1.50 \quad \$1.50 \quad \$1.50 \quad \$1.50$$
$$\$1.50 \quad \$1.50 \quad \$1.50 \quad \$1.50$$

The money/groups formulation will give students who struggle with the standard algorithm an alternative, but this thinking also helps students who prefer that more traditional approach. When a student sticks to the standard algorithm, there will be a time when the decimal point must be placed. The student will have four choices:

$$.120 \qquad 1.20 \qquad 12.0 \qquad 120$$

A student who has "eight groups of $1.50" echoing in their head, even if they do not do the addition, will be more likely to catch an error in decimal placement because three of the four possibilities are so obviously wrong.

There are other ways to think of this same multiplication, including "one-and-a-half groups of 8". Explore this approach as well. Do not write ½ as a fraction at this point — write the word "half". Ask what one-and-a-half groups of 8 means, and show one group of 8 and half a group of 8.

And yes, not all decimal products can be calculated in this way (think of $6.05 \cdot 1.6$), but estimation can often be used in these situations to continue to give students a good idea of a reasonable result.

You must carefully introduce the dimes/pennies and multiplication-as-groups formulations here because these ideas are going to reappear throughout the course.

After going over the puzzle with 8 and 1.5, give students time to continue working on the puzzles and apply these ideas about money and groups.

The type of puzzle to the right (where no bases are given) foreshadows a skill students will need near the end of this course when they factor trinomials. Avoid mentioning any of this to your students. We hope they will discover this connection themselves later on.

Many math teachers will solve this puzzle by focusing on the product of 24 because that involves fewer pairs to test (4) than the 8 whole-number pairs whose sum is 14. Teachers often decide very quickly to use this strategy (perhaps not realizing they are even making a decision), while some students may instead test pairs whose sum is 14. There is nothing wrong with this method, and students who proceed in this way should not be discouraged. In the class discussion of this example, though, draw out student strategies so that both methods can be described. Students themselves will probably notice that one method includes fewer options.

Students can be weak in organizing their work/thinking, and so when you see a student who is methodically writing down pairs of numbers that they are testing, praise that so that other students can hear. More typically, a student will write down a pair and then erase it when it does not solve the puzzle. Encourage students to keep a list and cross out the pairs that do not work, both to keep track of what they already tried, and also because later on it gives you and the student important

information on what was done. Math handouts covered in student work may not be attractive from a student's point of view, but they are very valuable to a teacher who is curious about understanding student thinking and to the student who may later forget what they did and thought.

[Not shown: "Core Math Expectations and the CUNY Start Math Binder"]

- **Uses of Parentheses**

  Your students should return to their notes. Ask them the following:
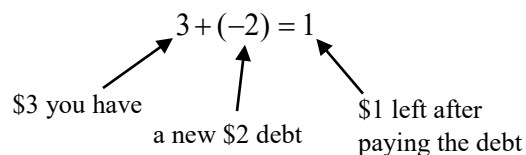
  What can parentheses indicate in math? (or in math problems?)

  As they offer ideas, encourage them to give a mathematical example as well as describe the use in words. At least the first three uses below should make it onto the board and into their notes. The fourth is not critical now, but if it comes up, that is fine. We will speak much more about ordered pairs later — when you reach ordered pairs in class 6, you can revisit these uses of parentheses.

  1. *Grouping* — we use parentheses to group numbers as a way of influencing the order of operations. *Ex.*: $7-(2+3)$, $10\times(6+3)$, or $20\div(4+1)$

  2. *Multiplication* — we use parentheses to indicate multiplication. *Ex.*: (5)(3) or 4(3)

  3. *Separation* — we use parentheses to separate a number's sign from an operation. *Ex.* $3+(-2)$

  4. *Ordered pair* — we use parentheses to indicate the coordinates of an ordered pair. *Ex.*: (2, 3)

- **Signed Numbers I: Addition**

  Fluency with signed numbers is very important in CUNY Start Math. Because our students first learned about signed numbers in various classrooms and countries with different methods and varied educational interruptions, it is impossible for us to reinforce each student's signed number arithmetic using the method they each first learned. Our only real choice is to carve a way of thinking about signed numbers that we will use in this course. Certainly, students can also use other methods. We can tell them that mathematicians need to be flexible enough to think about mathematical ideas in more than one way. We ask here that they develop a way of talking about signed-number addition in terms of "money held" and "debts" as the primary framework for building concrete conceptual understanding around adding signed numbers.

Via questions, facilitate a short all-class discussion in order to draw out this idea of what signed numbers can represent in terms of money and to establish some common language and thinking. Asking about specific bank balances can help — for instance, "*If your bank balance is –2, what does that mean?*" See the example to the right.

$$3 + (-2) = 1$$

$3 you have   a new $2 debt $1 left after paying the debt

Occasionally point to the parentheses and ask what they indicate (in this instance, separation).

Put a few more examples on the board and ask students to determine and explain each sum using the language of money and debts. Include an example with a negative and then a positive (varying the order from the initial example), an example with two negatives, and a third with two positives.

When you first introduce an example with two negatives, such as $-3 + (-9)$, at least one of your students will likely object to a result of –12 with the following:

"Wait — isn't negative and negative a positive?"

Even if nobody raises this idea, some of your students are probably thinking it. Do not discuss or try to teach multiplication rules here. We will look at multiplication much more carefully later. For now, gently suggest to the student that they are remembering a rule for a different situation (a different type of problem), and ask them not to rely here on any rule, but instead to focus on what makes sense using money and debts. When students think of this sum this way (as one $3 debt and another $9 debt), it will make sense that the result is negative.

Do not lecture about signed-number addition rules here or even draw them out. We do ask students to make their own generalization in the handout, but in general, our focus is on them using the money context as a way of imagining and calculating these sums. This money context gives students more conceptual understanding than rules such as "for same signs, add and keep the sign." Those mechanical procedures are easily jumbled in students' minds and then misused.

Give out the handout. As students are working on it, circulate and ask as many students as possible to describe a problem using the language of money and debts. Gradually, decimals appear in the examples. Hopefully the work done with the *Number Puzzles* will lead to strong student work here. Ask questions to reinforce what each digit represents in terms of money.

When discussing #2, after getting examples from students, ask if anyone knows a name for these pairs of numbers. Have students write "**opposites**" by these pairs on the handout. To reinforce this, quickly ask about the opposite of a few numbers. (*What's the opposite of 32? The opposite of –13?*) **What do these problems show us about opposites?** (Opposites always add up to zero.)

The problem asking students to generalize about the sum of two negatives is important in several respects. It is the first time we are asking students to write about their reasoning. We provide enough lines and space that students should realize we are looking for more than a one-word answer here. Give students time in pairs to discuss and respond thoughtfully to this. If you treat this problem superficially or permit them to write little or nothing, you are signaling to them that writing will not be valued in the course. That is not the tone we want to set at the outset. Some

students will write only about a specific example they provide, and in this instance you should press them to respond as generally as the prompt. A model response will look something like this:

> *No, the sum of two negative numbers is never positive. When a person has one debt and then takes on another debt, they will be further in debt. Debts are always negative.*

It is useful and clarifying for you and your students to get in the habit of saying "negative two" for the number –2, rather than "minus two". Especially when we work with subtraction in the next class, it will help to clarify when we are talking about a sign versus an operation. Certainly in the public schools, it is currently more in fashion to refer to the sign of –2 strictly as "negative" and not as "minus" as was more common a few decades ago. Some professors may still say "minus two", though, and so an explicit conversation about this is probably a good idea.

- **Best Buy Commissions**

  Give out the handout. It is fine to have calculators around for this activity.

  Once students have completed the table, discuss this job and their solution methods. For the trickier ones where the weekly pay is given, some students work backwards using inverse operations to determine the missing numbers of smartphones, while others use trial and error, perhaps using information already in the table to guide their trial and error. Highlight all of these legitimate methods in order to set the tone early on that there is often more than one correct way to solve a problem. If you tell students they should use inverse operations, they may believe you, but it will not make sense to some of them. Working with inverse operations is not the objective of this activity, and you do not want to get off track here. The important thing is for students to be able to complete the table with or without a calculator using whatever method works for them.

  Once you have discussed their solution methods, turn the sheet over and have them recopy the weekly pay for the given inputs. They should leave the center column blank until you are ready to move forward together.

Starting with 2 smartphones sold (not 0 or 1 because they are a bit special), ask a student to remind you how they calculated the weekly pay. Write the calculation in the center column.

Also do this for 10 phones and 16 phones. It can be okay if they start the calculations with the 150, except that it will introduce an order of operations issue. Once you have an expression in the middle column, try to remain consistent.

Smartphones You Sell in 1 Week		Your Weekly Pay
0		150
1		168
2	$2 \times 18 + 150$	186
10	$10 \times 18 + 150$	330
16	$16 \times 18 + 150$	438

Ask students to describe, in general, what you can do to the number of smartphones to determine the weekly pay.

35

Once a general pattern has been described, ask if it holds when the number of smartphones is 0 or 1. Some will not realize the multiplication still applies in the cases of 0 and 1, and it is important for them to see that they still fit the pattern. Complete the table in this way.

Smartphones You Sell in 1 Week		Your Weekly Pay
0	$0 \times 18 + 150$	150
1	$1 \times 18 + 150$	168
2	$2 \times 18 + 150$	186
10	$10 \times 18 + 150$	330
16	$16 \times 18 + 150$	438

Once you have discussed the pattern and written "**X 18 then +150**" in the second row of the table, ask your students if they know what words mathematicians use to describe our "starting" numbers, the "ending" numbers, or the repeated set of operations that we used to get from one to the other. When they or you introduce these terms, add the titles that you see in the first row of the table.

See the completed table at the right as a model.

Inputs	Function Rule	Outputs
Smartphones You Sell in 1 Week	**X 18 then +150**	Your Weekly Pay
0	$0 \times 18 + 150$	150
1	$1 \times 18 + 150$	168
2	$2 \times 18 + 150$	186
10	$10 \times 18 + 150$	330
16	$16 \times 18 + 150$	438

We do not need a formal definition of function here — please do not mention the vertical line test or formal characteristics such as "each and every input is associated with exactly one output". Our goal here is to give a friendly idea of a function as a set of starting numbers (inputs) that all go through the same consistent rule to produce a set of ending numbers (outputs).

Do <u>not</u> introduce variables or equations at this stage. These are going to be phased in across subsequent classes.

Ask if there could be other inputs. Why? Push students to explain in terms of the job. Solicit one or two examples and add them to the bottom of the table. Ask if zero is the number of smartphones sold the first week, one the number sold in the second week, and so on. Try to elicit the conclusion that these inputs and outputs represent just possibilities, not actual results for the first few weeks.
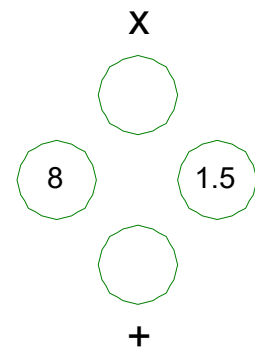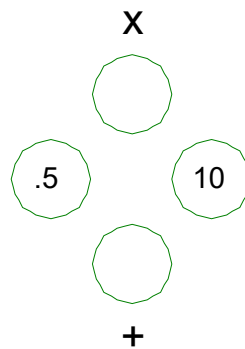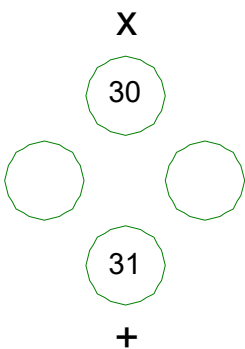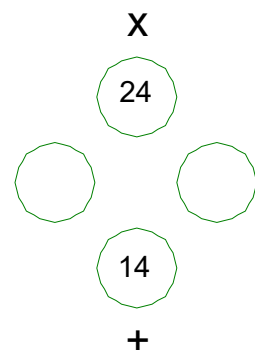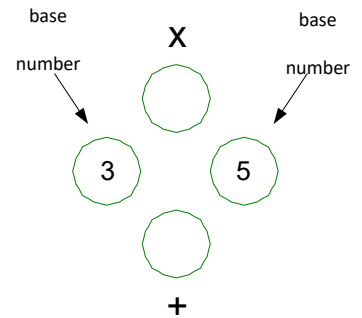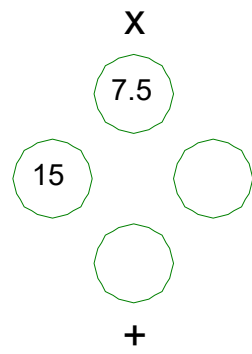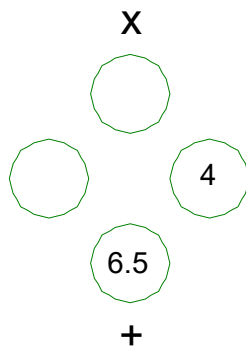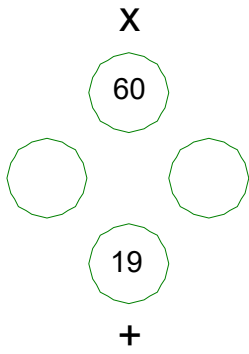
B.1.2 Class 1 Handouts

Number Puzzles

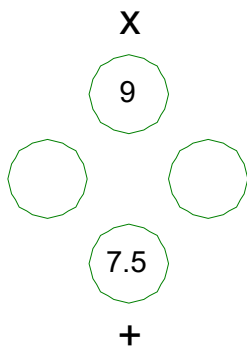*Adapted from College Preparatory Mathematics*

A *Number Puzzle* always includes four circles.
The circles on the left and right are our two "base" numbers.
The top circle is for the *product* of the two base numbers.
The bottom circle is for the *sum* of the two base numbers.

In each *Number Puzzle*, you are given two numbers. Your task is to
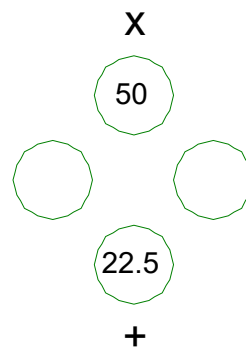figure out the missing numbers. Good luck!

base   X   base

number   number

3   5

+

X

18

3

+

X

3

3

+

X

24

14

+

X

30

31

+

X

.5   10

+

X

8   1.5

+

X

60

19
+

X

4

6.5
+

X

7.5

15
+

Challenge!
X

9

7.5
+

Challenge!
X

50

22.5
+

Signed Numbers I: Addition

1. Calculate each sum, and fill in the blanks to make true equations.

   a. $4 + (-2)$

   b. $-4 + 5$

   c. $-3 + (-4)$

   d. $-5 + 3$

   e. $.3 + (-.24)$

   f. $-1 + \underline{\quad} = -6$

   g. $-1 + (-2) + 5$

   h. $4 + (-3) + 1$

   i. $\underline{\quad} + 100 = 96$

   j. $-100 + (-50)$

   k. $-2.5 + 6$

   l. $-20 + (-5)$

2. Provide a **different** pair of numbers to complete each equation. At least one pair must include a decimal number.

   a. $\underline{\quad} + \underline{\quad} = 0$

   b. $\underline{\quad} + \underline{\quad} = 0$

   c. $\underline{\quad} + \underline{\quad} = 0$

3. Create three of your own signed number addition problems, and then solve them. At least one of your problems must include a decimal. Challenge yourself!

    a.   b.   c.

4. Can the sum of two negative numbers ever be positive? Defend your answer with at least one example and a written explanation using the language of "money" and "debts."

    _____

    _____

    _____

    _____

    _____

    _____

5. Calculate each sum, and fill in the blank to make a true equation.

   a. $10 + (-14)$ b. $1{,}000 + (-2)$ c. $8 + 40$ d. $-4 + (-40)$
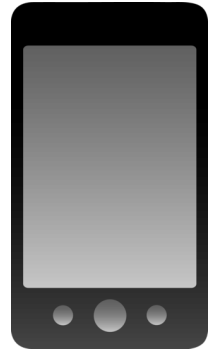
   e. $7.2 + (-7.2)$ f. $\_\_\_ + (-40) = 10$ g. $-4.5 + 2.5$ h. $-6.25 + 3.5$

Best Buy Commissions

You take a job at *Best Buy* selling smartphones. Your base pay is $150 per week. For each smartphone that you sell, you earn an additional $18.

Complete the table.

Smartphones You Sell in One Week	Your Weekly Pay
0	
1	
2	
10	
16	
	$366
	$492
	$546

Smartphones You Sell in One Week		Your Weekly Pay
0		
1		
2		
10		
16		

B.2.1 Class 7 Instructional Notes

[Not Shown: "Review Extra Practice #3"]

- **Counting Cubes**

  Rather than starting by defining the meaning of an exponent, this functions activity will give your students a chance to reveal their prior knowledge if they have it. It also can be used to help them understand (for many, for the first time) why we can say "five squared" in place of "five to the second power" and "two cubed" in place of "two to the third power."

  You will need a set of 1-inch foam cubes and a single ruler for this activity. Before giving out the handout, take 36 one-inch cubes and arrange the three figures on a central table. It is best to label them "figure 1", "figure 2", and "figure 3" using index cards in order to correspond to the handout.

  Have your students come around and look at what you have done. Ask them to describe what they see. You will likely hear "cube" and "square" among the replies. Put these words on the board as they come up, and try to have students clarify what they mean. Elicit clarity that these figures are not "squares" — squares are flat and only exist in flat surfaces.

  Ask students for their observations about a cube. Each cube has six square "faces". You may introduce the word "edge" to describe the segments that connect the faces. In trying to describe the pattern, students may say "one, two, three." If this happens, question them until they make and agree on more precise statements. It would be good to have a ruler there so that they can make some statements about length. And while in theory it might be nice to discuss area and volume, it is probably too time-consuming at this moment, and it could lead students to try to use formulas to solve the upcoming problem rather than exploring, coming up with their own approaches, and collaboratively making sense of it in their own ways.

  In this discussion before you give out the handout, do not bring out the number of cubes in Figure 2 or Figure 3. If students start to raise that, try to cut off that line of conversation. Do not let them explain how they calculated the number of cubes in any of the figures. That would give away far too much for this activity, which is designed for students to work in groups to figure out how many cubes are in each figure and why. We should expect and allow students to struggle with

this. A broad central goal of this activity is to help develop students' persistence, ability to collaborate, and inclination to try things and develop their own problem-solving strategies.

After this initial discussion, form groups of about four students and give each student the handout. We suggest that you lay it face-up on the desk and not remark on the questions on the back. Students may solve problem #1 using a variety of methods, and we would not want to indicate to them at the outset that they should use a function.

This should be an interesting challenge for your students. Some will try to draw the figure, but this is very difficult to do on a flat surface. Others may try and build the figure, and you should certainly offer cubes to each group in case they wish to do so. The only problem is that there are not enough cubes to build the figure once, let alone in each group. This false offering gently forces them to search for another solution method.

Frustration can set in with this problem. That can make this a good opportunity to talk about a growth mindset and the article about how our brains can grow like muscles. Working on this problem may not be comfortable for students, but that is because they are being challenged here. You can discuss how they are stretching their brains and growing their intelligence.

One reason why there can be a lot of confusion within groups is because students are trying to articulate their ideas without being able to build the 8th figure. A reasonable suggestion you can make is to ask students to try and solve an easier problem — the number of cubes in the fourth figure. This allows them to at least partially build the figure — enough to visualize it. Typically, once they continue the pattern for one more figure, they can extend that reasoning to the 8th figure.

When groups solve #1 and move on to making a table, push them to label the columns more clearly than just as "input" and "output" or x and y. They can label the columns in those ways, but we want them to also indicate more specifically what the columns represent in this context.

For the function rule in #3, there are a few possible answers:

$$y = x \cdot x \cdot x \qquad\qquad y = x^2 \cdot x \qquad\qquad y = x^3$$

The middle function equation is actually quite common because students may count the cubes by first figuring out the number on the top layer and then multiplying by the number of layers. $x^2 \cdot x$ flows directly from that thinking.

Encourage the different function equations firstly because they are all correct, but more importantly because they allow students to demonstrate exponent notation.

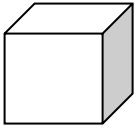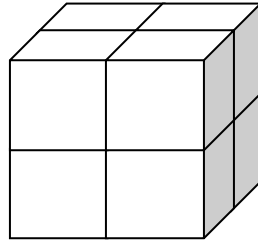**[Not Shown: "Exponents I"]**
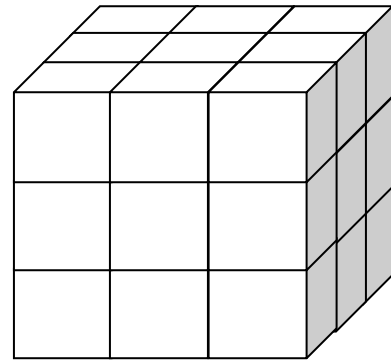
Counting Cubes

Figure 1

Figure 2

Figure 3

1. Determine the number of small cubes that will be included in the 8th figure. Be prepared to explain how your group determined your answer.

2. Create a table of values where the input is the figure number (1, 2, 3, etc.), and the output is the number of small cubes needed to make that figure. Label the table clearly.

3. Determine a function rule that describes the relationship between inputs and outputs.

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